Multi-Parametric Nonlinear Program

Minghao Mou

March 28, 2024

Outline

The Problem I

$P(\epsilon)$:

$$\min_{x} \ f(x,\epsilon), \tag{1a}$$

$$\text{s.t. } x \in R(\epsilon), \tag{1b}$$

where $\epsilon \in T \subset \mathbb{R}^r$, $R : \epsilon \mapsto 2^{\mathbb{R}^n}$.
$P_1(\epsilon)$:

$$\min_{x} \ f(x,\epsilon), \tag{2a}$$

$$\text{s.t. } x \in R(\epsilon) := \{x \in M : g(x,\epsilon) \leq 0, h(x,\epsilon) = 0\}, \tag{2b}$$

where $M \subset \mathbb{R}^n$, $g : \mathbb{R}^n \times T \to \mathbb{R}^p$, and $h : \mathbb{R}^n \times T \to \mathbb{E}^q$.
$P_0(\epsilon)$:

$$\min_{x} \ f(x,\epsilon), \tag{3a}$$

$$\text{s.t. } x \in M, \tag{3b}$$

Preliminaries

Definition (Locally Lipschitz)

A function $\phi : \mathbb{R}^r \to \mathbb{R}^l$ is said to be *locally Lipschitz* near $\bar{z} \in \mathbb{R}^r$ if there is a neighborhood $N(\bar{z})$ of $\bar{z}$ and a $L > 0$ such that

$$\|\phi(z_1) - \phi(z_2)\| \leq L\|z_1 - z_2\|, \quad \forall z_1, z_2 \in N(\bar{z}). \tag{4}$$

Definition (Directional Derivative)

The *directional derivative* of ϕ at the point $\bar{z}$ in the direction $v \in \mathbb{R}^r$ is defined as

$$D\phi(\bar{z}; v) := \lim_{\beta \to 0^+} \frac{1}{\beta} \left[\phi(\bar{z} + \beta v) - \phi(\bar{z})\right]. \tag{5}$$

If the above limit exists for every $v \in \mathbb{R}^r$, we say ϕ is *directionally differentiable* at $\bar{z}$. Directionally differentiability does not imply differentiability.

Continuity I

In this section we will consider conditions under which the optimal-value function $f^\star$ and the optimal solution map S are continuous. We begin with definitions of continuity of a point-to-set map. Let T be a metric space, $\psi : T \to 2^{\mathbb{R}^n}$, and $\bar{\epsilon} \in T$.

Definition (Semi-continuity)

A point-to-set function ψ is said to be *upper (lower) semi-continuous* at $\bar{\epsilon}$ if for each open set $O \subset \mathbb{R}^n$ satisfying $\psi(\bar{\epsilon}) \in O$ $(\psi(\bar{\epsilon}) \cap O \neq \emptyset)$, there exists a neighborhood $N(\bar{\epsilon})$ such that $\psi(\epsilon) \subset O$ $(\psi(\epsilon) \cap O \neq \emptyset)$, for all $\epsilon \in N(\bar{\epsilon})$.

Definition (Closedness)

A point-to-set function ψ is said to be *closed* at $\bar{\epsilon}$ if $\epsilon_n \in T, \epsilon_n \to \bar{\epsilon}, x_n \in \psi(\epsilon_n)$, and $x_n \to \bar{x}$ imply $\bar{x} \in \psi(\bar{\epsilon})$.

Continuity II

Definition (Openness)

A point-to-set function ψ is said to be *open* at $\bar{\epsilon}$ if $\epsilon_n \in T, \epsilon_n \to \bar{\epsilon}$, and $\bar{x} \in \psi(\bar{\epsilon})$ imply that $\exists m$ and $\{x_n\}$ such that $x_n \in \psi(\epsilon_n)$ for all $n \geq m$ and $x_n \to \bar{x}$.

Definition (Uniform Compactness)

A point-to-set function ψ is said to be *uniformly compact* near $\bar{\epsilon}$ if the set $\cup_{\epsilon \in N(\bar{\epsilon})} \psi(\epsilon)$ is bounded for some neighborhood $N(\bar{\epsilon})$.

Definition

Let $\{A_n\}$ be a sequence of subsets of $\mathbb{R}^n$. The *inner limit* of $\{A_n\}$ is defined as

$$\underline{\lim}_{n\to\infty} A_n := \{x \in \mathbb{R}^n : \exists m, \{x_n\} \text{ such that } x_n \in A_n, \forall n \geq m, x_n \to x\}. \quad (6)$$

Hogan showed that (a) lower semicontinuity and openness at a point are equivalent, and (b) if ψ is uniformly compact near $\bar{\epsilon}$, then ψ is closed if and only if $\psi(\bar{\epsilon})$ is compact and ψ is upper semicontinuous at $\bar{\epsilon}$.

Continuity III

Lemma

ψ is closed (open) at $\bar{\epsilon}$ if and only if $\underline{\lim}_{n\to\infty}\psi(\epsilon_n) \subset (\supset)\psi(\bar{\epsilon})$, for any $\{\epsilon_n\} \subset T$ such that $\epsilon_n \to \bar{\epsilon}$.

Definition (Continuity)

ψ is *continuous at* $\bar{\epsilon}$ if it is upper semicontinuous and lower semicontinuous at $\bar{\epsilon}$.

Theorem (Theorem 2.1 in [FI90])

For $P(\epsilon)$, we have
(a). If R is lower semicontinuous at $\bar{\epsilon}$, and f is usc on $R(\bar{\epsilon}) \times \{\bar{\epsilon}\}$, then $f^\star$ is usc at $\bar{\epsilon}$.
(b). If R is upper semicontinuous at $\bar{\epsilon}$, $R(\bar{\epsilon})$ is compact, and f is lsc on $R(\bar{\epsilon}) \times \{\bar{\epsilon}\}$, then $f^\star$ is lsc at $\bar{\epsilon}$.

Continuity IV

Theorem (Theorem 2.2 in [FI90])

For $P(\epsilon)$, if
(1). f is continuous on $R(\bar{\epsilon}) \times \{\bar{\epsilon}\}$;
(2). R is closed and open (i.e. lower semi-continuous) at $\bar{\epsilon}$;
(3). $S(\bar{\epsilon})$ is nonempty and a singleton;
(4). S is uniformly compact near $\bar{\epsilon}$;
then S is closed and open (under (4), S is continuous) at $\bar{\epsilon}$.

Note that (1) openness is equivalent to lower semi-continuity, and (2) uniform compactness + closedness imply upper semi-continuity. Therefore, S is continuous.

Continuity V

Theorem (Theorem 2.3 in [FI90])

For $P(\epsilon)$, if
(1). f is quasi-convex in x for every fixed $\epsilon \in T$ and continuous on $\mathbb{R}^n \times T$;
(2). R is closed at every ϵ near $\bar{\epsilon}$ and open at $\bar{\epsilon}$.
(3). R is convex-valued near $\bar{\epsilon}$ (i.e. $R(\epsilon)$ is convex near for ϵ near $\bar{\epsilon}$);
then $S(\epsilon)$ is nonempty and uniformly compact near $\bar{\epsilon}$ if and only if $S(\bar{\epsilon})$ is nonempty and compact.

Theorem (Theorem 2.4 in [FI90])

For $P(\epsilon)$, if
(1). $f(x, \epsilon) = \max\{f_1(x, \epsilon), f_2(\epsilon)\}$, where f_1 is continuous on $\mathbb{R}^n \times T$ and strictly quasi-convex in x for each fixed $\epsilon \in T$, and f_2 is continuous on T;
(2). R is nonempty, convex-valued, and continuous on T;
then S is continuous and convex-valued on T.

In order to specialize these general theorems to the concrete nonlinear (or linear) programming problem $P_1(\epsilon)$, we need to know the conditions that guarantee the

Continuity VI

continuity of an inequality and/or equality constraint map R of $P_1(\epsilon)$. The following two theorems are concerned with the constraint map:

$$R(\epsilon) := \{x \in M : g(x) \leq \epsilon\}, \tag{7}$$

where $M \subset \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^p$, and $\epsilon \in \mathbb{R}^p$.

Theorem (Theorem 2.5 in [FI90])

For the map R of (7), suppose that $M = \mathbb{R}^n$, g is continuous on $\mathbb{R}^n$, and that $R(\bar{\epsilon})$ is compact. Then
(1). R is upper semi-continuous at $\bar{\epsilon}$ if and only if there exists a vector $\epsilon' > \bar{\epsilon}$ such that $R(\epsilon')$ is a compact set.
(2). if the set $R^0(\bar{\epsilon}) := \{x \in \mathbb{R}^n : g(x) < \bar{\epsilon}\}$ is nonempty, then R is lower semi-continuous at $\bar{\epsilon}$ if and only if $\overline{R^0(\bar{\epsilon})} = R(\bar{\epsilon})$, namely,

$$\overline{\{x \in \mathbb{R}^n : g(x) < \bar{\epsilon}\}} = \overline{R^0(\bar{\epsilon})} = R(\bar{\epsilon}) = \{x \in \mathbb{R}^n : g(x) \leq \bar{\epsilon}\}$$

Continuity VII

Note that in (1) of the above theorem, the compactness of $R(\bar{\epsilon})$ and the upper semi-continuity of R at $\bar{\epsilon}$ imply the closedness of R at $\bar{\epsilon}$.

Theorem (Theorem 2.6 in [FI90])

For the map R of (7), suppose that M is compact and convex, and that g_i are lsc and strictly convex on M. Then R is closed (i.e. upper semi-continuous, under the assumptions) and open (i.e. lower semi-continuous) at every $\epsilon \in \mathrm{dom}(R)$ relative to $\mathrm{dom}(R)$, where $\mathrm{dom}(R) := \{\epsilon \in \mathbb{R}^p : R(\epsilon) \neq \emptyset\}$.

The next theorem, dut to Dantzig et al., is concerned with a linear inequality (and equality) constraint:

$$R(A, b) := \{x \in M : Ax \geq b\}, \tag{8}$$

where $M \subset \mathbb{R}^n$, A is a $p \times n$ real matrix, and $b \in \mathbb{R}^p$.

Continuity VIII

Theorem (Theorem 2.7 in [FI90])

Let $\bar{A}$ *be a* $p \times n$ *real matrix consisting of* p *row vectors* $\bar{a}_i^\top \in \mathbb{R}^n, i = 1, ..., p$, $\bar{b} \in \mathbb{R}^p$, *and*

$$I := \{i = 1, ..., p : \bar{a}_i^\top x = \bar{b}_i, \forall x \in R(\bar{A}, \bar{b})\}.$$

If the matrix $\bar{A}_I$*, whose rows are* $\bar{a}_i^\top, i \in I$*, has full rank, then, for every sequence* $\{(A_n, b_n)\}$ *converging to* $(\bar{A}, \bar{b})$*, either*

$$\underline{\lim}_{n\to\infty} R(A_n, b_n) = R(\bar{A}, \bar{b}),$$

or $R(A_n, b_n)$ *is empty for infinitely many* n*, and consequently,* R *is closed and open at* $(\bar{A}, \bar{b})$ *relative to* $\mathrm{dom}(R)$ *if* $(\bar{A}, \bar{b}) \in \mathrm{dom}(R)$.

We next consider nonlinear inequality constraints. Let

$$R(\epsilon) := \{x \in M : g(x, \epsilon) \le 0\}. \tag{9}$$

Continuity IX

Theorem (Theorem 2.8 in [FI90])

For the map R of (9), suppose that M is closed, g is continuous on $M \times \{\bar{\epsilon}\}$, and that

$$\overline{\{x \in M : g(x, \bar{\epsilon}) < 0\}} = R(\bar{\epsilon}),$$

then R is closed and open at $\bar{\epsilon}$.

Theorem (Theorem 2.9 in [FI90])

For the map R of (9), suppose that M is compact and convex, g is continuous on $M \times T$, and that g_i are strictly convex on M for each fixed $\epsilon \in T$. Then R is closed and open at every $\epsilon \in \mathrm{dom}(R)$ relative to $\mathrm{dom}(R)$.

By using implicit function theorem, Aiyoshi extends Theorem 2.8 to the inequality-equality constrained case.
Stein-Topkis essentially show that, for $P(\epsilon)$, if f and R are locally Lipschitz (in the sense of Hausdorf distance), then $f^\star$ is locally Lipschitz. For the inequality-equality constrained problem $P_1(\epsilon)$, some **constraint qualifications**

Continuity X

(e.g. MFCQ and LICQ) and the uniform compactness of R are sufficient for $f^\star$ to be (locally Lipschitz) continuous.

Convexity of Optimal-value Function I

Throughout this section it is assumed that $T \subset \mathbb{R}^r$ is nonempty and convex.

Definition (Convex Map)

A point-to-set map $R : T \to 2^{\mathbb{R}^n}$ is said to be *convex (concave)* on T if for all $\epsilon_1, \epsilon_2 \in T$ and $\lambda \in (0, 1)$

$$\lambda R(\epsilon_1) + (1 - \lambda)R(\epsilon_2) \subset (\supset) R\left(\lambda\epsilon_1 + (1 - \lambda)\epsilon_2\right). \tag{10}$$

- If in (10), the inclusion $\subset$ holds only for all $\epsilon_1 \neq \epsilon_2 \in T$ and $\lambda \in (0, 1)$, then R is said to be *essentially convex* on T.
- R is said to be *essentially affine* on T if R is both essentially convex and concave on T.

Convexity of Optimal-value Function II

Theorem (Theorem 3.1 in [FI90])

For $P(\epsilon)$, suppose that f is jointly convex on $\{(x, \epsilon) : x \in R(\epsilon), \epsilon \in T\}$, and that R is essentially convex on T. Then $f^\star$ is convex on T.

Definition (Quasi-convexity)

A function $\phi : M \to \mathbb{R}$ is said to be (strictly) quasi-convex on a convex set M if, for all $x_1, x_2 \in M$ and $\lambda \in (0, 1)$

$$\phi(\lambda x_1 + (1 - \lambda)x_2) \leq (<) \min\{\phi(x_1), \phi(x_2)\}. \tag{11}$$

Theorem (Theorem 3.2 in [FI90])

For $P_1(\epsilon)$, suppose that g_i are jointly quasi-convex on $M \times T$, h_j are jointly affine on $M \times T$, and that M is convex. Then R is convex on T.

Convexity of Optimal-value Function III

Theorem (Theorem 3.3 in [FI90])

For $P(\epsilon)$, suppose that f is jointly concave on $\mathbb{R}^n \times T$, and that R is concave on T. Then $f^\star$ is concave on T.

Theorem (Theorem 3.4 in [FI90])

For $P_0(\epsilon)$, suppose that f is concave in ϵ on T for all $x \in M$. Then $f^\star$ is concave on T.

Theorem (Theorem 3.5 in [FI90])

For $P(\epsilon)$, suppose that f is jointly affine on $\mathbb{R}^n \times T$, R is essentially affine on T, and that $T \subset \mathrm{dom}(R)$. Then $f^\star$ is both convex and concave on T, and S is essentially affine on T.

Differential Stability I

A number of results on the rate of change in $f^\star$ under small perturbations have been obtained by means of the first- and second-order directional derivatives of $f^\star$ in the direction along which the perturbation is made. The first result is due to Danskin.

Theorem (Theorem 4.1 in [FI90])

For $P_0(\epsilon)$, suppose that $T = \mathbb{R}^r$, and that f and $\nabla_\epsilon f$ are continuous on $M \times N(\epsilon)$, where $N(\epsilon)$ is a neighborhood of $\epsilon \in \mathbb{R}^r$. Then $f^\star$ is locally Lipschitz near ϵ, directionally differentiable at ϵ, and

$$Df^\star(\epsilon; v) = \min_{x \in S(\epsilon)} \nabla_\epsilon f(x, \epsilon) v. \tag{12}$$

Another theorem, dut to Gauvin-Dubeau and Fiacco, gives lower and upper bounds for the directional derivative of $f^\star$ in $P_1(\epsilon)$. In the sequel, we assume in

Differential Stability II

$P_1(\epsilon)$, $M = \mathbb{R}^n$, and that f, g, and h are continuously differentiable in (x, ϵ). The Lagrangian and the set of multipliers of $P_1(\epsilon)$ are defined as follows:

$$L(x, u, w, \epsilon) = f(x, \epsilon) - u^\top g(x, \epsilon) + w^\top h(x, \epsilon),$$
$$K(x, \epsilon) = \{(u, w) \in \mathbb{R}^p \times \mathbb{R}^q : \nabla_x L(x, u, w, \epsilon) = 0, u_i g_i(x, \epsilon) = 0, u_i \geq 0, i \in [p]\}. \tag{13}$$

Definition (Mangasarian-Fromovitz Constraint Qualification (MFCQ))

We say that MFCQ holds at $x \in R(\epsilon)$ for $P(\epsilon)$ if
(1). the vectors $\{\nabla_x h_j(x, \epsilon), j \in [p]\}$ are linearly independent;
(2). there exists $z \in \mathbb{R}^n$ such that

$$\begin{aligned} &\nabla_x g_i(x, \epsilon) > 0, i \in \mathcal{A}(x, \epsilon) \\ &\nabla_x h_j(x, \epsilon) = 0, \forall j \in [q], \end{aligned} \tag{14}$$

where $\mathcal{A}(x, \epsilon) := \{i \in [p] : g_i(x, \epsilon) = 0\}$.

Differential Stability III

Theorem (Theorem 4.2 in [FI90])

For $P_1(\epsilon)$, suppose that $M = \mathbb{R}^n$, that $R(\epsilon) \neq \emptyset$ and R is uniformly compact near $\epsilon \in \mathbb{R}^r$, and that (MFCQ) holds at each $x \in S(\epsilon)$. Then $f^\star$ is locally Lipschitz near ϵ, and for any $v \in \mathbb{R}^r$,

$$\begin{aligned}\inf_{x\in S(\epsilon)} \min_{(u,w)\in K(x,\epsilon)} \nabla_\epsilon L(x,u,w,\epsilon) &\leq \liminf_{\beta\to 0^+} \frac{1}{\beta}\left[f^\star(\epsilon+\beta v) - f^\star(\epsilon)\right] \\ &\leq \limsup_{\beta\to 0^+} \frac{1}{\beta}\left[f^\star(\epsilon+\beta v) - f^\star(\epsilon)\right] \\ &\leq \inf_{x\in S(\epsilon)} \max_{(u,w)\in K(x,\epsilon)} \nabla_\epsilon L(x,u,w,\epsilon)\end{aligned} \tag{15}$$

Furthermore, if f and g are convex on $\mathbb{R}^n \times \{\epsilon\}$, and if h is affine on $\mathbb{R}^n \times \{\epsilon\}$, then $f^\star$ is directionally differentiable at ϵ, and

$$Df^\star(\epsilon; v) = \min_{x\in S(\epsilon)} \max_{(u,w)\in K(\epsilon)} \nabla_\epsilon L(x,u,w,\epsilon)v, \tag{16}$$

where, under the assumptions $K(x,\epsilon) = K(\epsilon)$ is constant for $x \in S(\epsilon)$.

Differential Stability

There are two immediate consequences of Theorem 4.2. Under stronger CQ than (MFCQ) at $x \in S(\epsilon)$ (e.g. (SMFCQ), or (LICQ)), $K(x, \epsilon)$ reduces to a singleton, say $\{u(x), w(x)\}$, so that (16) reduces to

$$Df^{\star}(\epsilon; v) = \min_{x \in S(\epsilon)} \nabla_{\epsilon} L[x, u(x), w(x), \epsilon]v \tag{17}$$

Another special case is the jointly convex case: if f and g are jointly convex and h is jointly affine on $\mathbb{R}^n \times \mathbb{R}^r$, then for each $(u, w) \in K(x, \epsilon)$, $\nabla_{\epsilon} L(x, u, w, \epsilon)$ does not depend on $x \in S(\epsilon)$, and hence (16) becomes

$$Df^{\star}(\epsilon; v) = \max_{(u,w) \in K(\epsilon)} \nabla_{\epsilon} L(x, u, w, \epsilon)v, \tag{18}$$

where $x \in S(\epsilon)$. Since in this case, $f^{\star}$ is convex (since f is jointly convex and $R(\epsilon)$ is convex for all ϵ. See Theorem 3.1). The above equation means that for each $x \in S(\epsilon)$ and $(u, w) \in K(x, \epsilon)$, $\nabla_{\epsilon} L(x, u, w, \epsilon)$ is a subgradient of $f^{\star}$ at ϵ.

Reference

Anthony V Fiacco and Yo Ishizuka.
Sensitivity and stability analysis for nonlinear programming.
Annals of Operations Research, 27(1):215–235, 1990.