IE690 Course Notes*

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*[Under construction] This is the notes I took when taking the course *IE690: Optimization, Game Theory, and uncertainty*, taught by Prof. Andrew Liu in Spring 2024.

1 Convex Analysis

Definition 1.1 (Cone)**.** A nonempty set $S \subset \mathbb{R}^n$ is a *cone* if $x \in C$ implies $\alpha x \in C$ for all $\alpha > 0$.

Remark 1.2. A cone may not be convex/closed/bounded.

Theorem 1.3 (Closed-Point)**.** Let $S \neq \emptyset$, closed, and convex in $\mathbb{R}^n$. For any $y \notin S$, there exists a $\tilde{x} \in S$ such that

$$\|\tilde{x} - y\|_2 \leq \|x - y\|_2, \quad \forall x \in S.$$

We can the point $\tilde{x}(y)$ the projection of y onto S.

Theorem 1.4 (Separation of Convex Sets)**.** Let $S \subset \mathbb{R}^n$ be nonempty, closed, and convex. Let $y \notin S$, then there exists a $p \in \mathbb{R}^n$ and a $\alpha \in \mathbb{R}$ such that $p^{\mathrm{T}} y > \alpha$ and $p^{\mathrm{T}} x \leq \alpha, \forall x \in S$.

Theorem 1.5 (Supporting Hyperplane)**.** Let $S \subset \mathbb{R}^n$ be nonempty, closed, and convex. Then for any $\bar{x} \in \partial S$, there exists a $p \in \mathbb{R}^n$ such that the hyperplane $H := \{x : p^{\mathrm{T}}(x - \bar{x}) = 0\}$ supports S at $\bar{x}$, namely,

$$p^{\mathrm{T}}(y - \bar{x}) \geq (\leq) 0, \quad \forall y \in S.$$

Theorem 1.6. If $f : X \to \mathbb{R}$ is convex, then f is continuous on $\mathrm{int}(X)$.

Definition 1.7 (Subgradient)**.** Let $X \neq \emptyset$ and be convex in $\mathbb{R}^n$, and $f : X \to \mathbb{R}$ be convex. Then g is called a subgradient of f at $\bar{x} \in X$ if

$$f(x) \geq f(\bar{x}) + g^{\mathrm{T}}(x - \bar{x}), \quad \forall x \in X.$$

Definition 1.8 (Subdifferential)**.** The subdifferential of f at $\bar{x}$ is the collection of all subgradients of f at $\bar{x}$. Namely,

$$\partial f(\bar{x}) := \{g : f(x) \geq f(\bar{x}) + g^{\mathrm{T}}(x - \bar{x}), \forall x \in X\}.$$

Remark 1.9. $\partial f(\bar{x}) \neq \emptyset$ for all f.

2 Optimality Conditions

2.1 Convex Problem

Even though we have a convex objective and a convex set, we may still do not have existence of optimal solution.

Theorem 2.1 (Weierstrass)**.** For $X \subset \mathbb{R}^n$ nonempty and $f : X \to \mathbb{R}$ continuous. An optimal solution to $\min_{x \in X} f(x)$ exists if and only if one of the followings holds:

1. X is compact.
2. X is closed and f is coercive[1].
3. $\exists \gamma$ such that $\{x : f(x) \leq \gamma\}$ is nonempty and compact.

Lemma 2.2. A local minimizer to a convex program is also a global minimizer.

Lemma 2.3. If f is strictly convex, then there can be at most one optimal solution.

Theorem 2.4 (Optimality Condition)**.** If f is differentiable, $x^\star$ is an optimal solution if and only if

$$\nabla f(x^\star)^{\mathrm{T}}(x - x^\star) \geq 0, \quad \forall x \in X. \tag{1}$$

Remark 2.5. Equation (1) is actually the variational inequality $\mathrm{VI}(\nabla f, X)$.

2.2 General Unconstrained Problem

Assume that f is smooth enough.

With convexity. $x^\star$ is optimal if and only if $\nabla f(x^\star) = 0$.

Without convexity. First-order necessary condition: If $x^\star$ is a local minimizer, then $\nabla f(x^\star) = 0$. Second-order sufficient condition: If $\nabla f(x^\star) = 0$ and $\nabla^2 f(x^\star)$ is PD, then $x^\star$ is a local minimizer. Replace PD by PSD, we obtain the second-order necessary condition.

[1] f is said to be coercive if $\lim_{\|x\| \to +\infty} f(x) = \infty$.

General Non-linear Programming. f may be non-convex and X may not be convex either. Before presenting the optimality conditions, we will first introduce some new concepts. Actually, the optimality conditions derived will be very abstract and will not be helpful in finding optimal solutions.

Definition 2.6 (Cone of Feasible Directions)**.** For $X \neq \emptyset$ in $\mathbb{R}^n$ and let $\bar{x} \in \text{cl}(X)$.

$$\mathcal{F}_X(\bar{x}) := \{d \in \mathbb{R}^n | d \neq 0, \bar{x} + \tau d \in X, \forall \tau \in (0, \delta), \exists \delta > 0\},$$

is called the cone of feasible directions of X at $\bar{x}$.

Definition 2.7 (Cone of Descent Directions)**.** The cone of descent directions of $f : X \to \mathbb{R}$ at $\bar{x} \in X$ is

$$\mathcal{D}_X(\bar{x}) := \{d : f(\bar{x} + \tau d) < f(\bar{x}), \forall \tau \in (0, \delta), \exists \delta > 0\}.$$

The optimality condition for any general non-linear programs can be written as $\mathcal{F}_X(x^\star) \cap \mathcal{D}_X(x^\star) = \emptyset$, which simply means that *any feasible direction is not a descent direction.* Usually, it is difficult to characterize $\mathcal{F}_X(\cdot)$ and $\mathcal{D}_X(\cdot)$.

Lemma 2.8. Define $\mathcal{D}_0(\bar{x}) := \{d : \nabla f(\bar{x})^{\mathrm{T}} d < 0\}$. Then, $\mathcal{D}_0(\bar{x}) \subset \mathcal{D}_X(\bar{x})$.

Definition 2.9 (Tangent Cone)**.** The tangent cone of X at $\bar{x} \in X$ is defined as

$$\mathcal{T}_X(\bar{x}) := \left\{ d : d = 0 \text{ or } \exists \{x_k\}_k \subset X, x_k \neq \bar{x}, \forall k, x_k \to \bar{x}, \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \to \frac{d}{\|d\|} \right\}.$$

Lemma 2.10. $\mathcal{F}_X(\bar{x}) \subset \mathcal{T}_X(\bar{x})$.

Theorem 2.11 (Optimality Condition)**.** $x^\star$ is optimal if and only if $\mathcal{T}_X(x^\star) \cap \mathcal{D}_0(x^\star) = \emptyset$.

Proof. Let $d \in \mathcal{T}_X(x^\star)$. WLOG assume that $d \neq 0$. Otherwise it is trivial. Apply the definition of tangent cone to find a $x_k \to x^\star$. Apply first-order Taylor expansion to $f(x_k)$, centered at $x^\star$, one can prove that $\nabla f(x^\star)^{\mathrm{T}} d \geq 0$. Therefore $d \notin \mathcal{D}_0(x^\star)$. □

Remark 2.12. This result is non-trivial since $\mathcal{F}_X(x^\star) \cap \mathcal{D}_X(x^\star) = \emptyset$ does not imply $\mathcal{T}_X(x^\star) \cap \mathcal{D}_0(x^\star) = \emptyset$.

Let $X := \{x \in \mathbb{R}^n : g_i(x) \le 0, h_j(x) = 0, i = 1, 2, ..., m, j = 1, 2, ..., n\}$.

Definition 2.13 (Linearized Cone)**.**

$$\mathcal{L}_X(\bar{x}) := \{d : \nabla g_i(\bar{x})^{\mathrm{T}} d \le 0, \nabla h_j(\bar{x})^{\mathrm{T}} d = 0, i \in \mathcal{A}(\bar{x}), j = 1, 2, ..., n\},$$

where $\mathcal{A}(\bar{x}) := \{i \in [m] : g_i(\bar{x}) = 0\}$.

Lemma 2.14. $\mathcal{F}_X(\bar{x}) \subset \mathcal{T}_X(\bar{x}) \subset \mathcal{L}_X(\bar{x})$.

Consider an example

Example 2.15.

$$\min_{x_1, x_2} \; -x_1 - x_2, \tag{2a}$$

$$\text{s.t. } x_1^2 + x_2^2 \le 0, \tag{2b}$$

$$x_1 \ge 0, x_2 \ge 0. \tag{2c}$$

If one computes the $\mathcal{D}_0((0,0))$, linearized, and tanget cones at $(0,0)$, we have $\mathcal{L}_X((0,0)) \cap \mathcal{D}_0((0,0)) \ne \emptyset$. The so-called *constraint qualification* guarantees that $\mathcal{T}_X(\bar{x}) = \mathcal{L}_X(\bar{x})$ for locally optimal $\bar{x}$. Therefore, one can check $\mathcal{L}_X(\bar{x}) \cap \mathcal{D}_0(\bar{x})$ to determine the optimality. As another example, the constraints are equivalent to $x_1 = x_2 = 0$. In this case, we have $\mathcal{T}_X((0,0)) = \mathcal{L}_X((0,0)) \cap \mathcal{D}_0((0,0)) = \emptyset$.

Theorem 2.16 (First-Order Necessary Condition: Karush-Khun-Tucker (KKT) Condition)**.** Assume f, g_i, and h_j are all continusouly differentiable. If the optimization problem

$$\min_{x} \; f(x), \tag{3a}$$

$$\text{s.t. } g_i(x) \le 0, i \in [m] \tag{3b}$$

$$h_j(x) = 0, j \in [n], \tag{3c}$$

has $x^\star$ as its local minimizer and $\mathcal{T}_X(x^\star) = \mathcal{L}_X(x^\star)$, then $\exists \lambda \in \mathbb{R}^m$ and

$\mu \in \mathbb{R}^n$ such that

$$\begin{aligned}
&\nabla f(x^\star) + \sum_{i=1}^m \lambda_i \nabla g_i(x^\star) + \sum_{j=1}^n \mu_j \nabla h_j(x^\star) = 0, \\
&\lambda \geq 0, \mu \geq 0, \\
&\lambda^{\mathrm{T}} g(x^\star) = 0, \\
&\mu^{\mathrm{T}} h(x^\star) = 0, \\
&g(x^\star) \leq 0, \\
&h(x^\star) = 0.
\end{aligned} \tag{4}$$

Remark 2.17. The proof of the theorem uses the famous *Farkas' Lemma.* We state the lemma here: Exactly one of the two linear systems of equations

$$\begin{cases} A^{\mathrm{T}} x \leq 0, \\ c^{\mathrm{T}} x > 0 \end{cases}, \quad \begin{cases} A^{\mathrm{T}} y = c, \\ y \geq 0, \end{cases}$$

has a solution.

2.3 More on Constraint Qualification

(1) Guiguard CQ. $\mathcal{T}_X^D(\bar{x}) = \mathcal{L}_X^D(\bar{x})$ (dual cone).

2) Abadie CQ. $\mathcal{T}_X(\bar{x}) = \mathcal{L}_X(\bar{x})$.

3) MFCQ. $\exists d$ such that $\nabla g_i(\bar{x})^{\mathrm{T}} d < 0, \forall i \in \mathcal{A}(\bar{x})$, $\nabla h_j(\bar{x})^{\mathrm{T}} d = 0$, and $\nabla h_j(\bar{x})$'s are linearly independent for all $j \in [n]$. MFCQ is equivalent to the boundedness of Lagrange multipliers.

4) LICQ. $\{\nabla g_i(\bar{x}), i \in \mathcal{A}(\bar{x}), \nabla h_j(\bar{x}), \forall j \in [n]\}$ are linearly independent, which is equivalent to the uniqueness of Lagrange multipliers.

5) Slater's CQ. g_i are all convex for all i and $\exists x_0 \in X$ such that $g_i(x_0) < 0$ for all i.

6) LCQ. All constraints are linear.

Implication: $4) \to 3) \to 2) \to 1); 6) \to 2)$.

3 Game Theory

3.1 Existence of Nash Equilibrium

Definition 3.1 (Normal Form)**.** The normal form of a game is $\{N, \{X_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N\}$, where N is the number of players, X_i is the action space of the i-th player, and $\pi_i : \prod_{i=1}^N X_i \to \mathbb{R}$ is the utility function of the i-th player.

Definition 3.2 (Complete Information)**.** A game is said to be of complete information if the normal form is known to all players.

Definition 3.3 (Nash Equilibrium)**.** The action profile $(x_1^\star, x_2^\star, ..., x_N^\star) \in \prod_{i=1}^N X_i$ is said to be a (pure-startegy) Nash equilibrium if

$$\pi_i(x_i^\star, x_{-i}^\star) \geq \pi_i(x_i, x_{-i}^\star), \quad \forall x_i \in X_i. \tag{5}$$

Definition 3.4 (Mixed-Strategy Nash Equilibrium)**.** A mixed-strategy profile $(\sigma_1^\star, ..., \sigma_N^\star) \in \prod_{i=1}^N \Delta(X_i)$ is a mixed-strategy Nash equilibrium of $G := \{N, \{X_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N\}$ if for all i,

$$\pi_i(\sigma_i^\star, \sigma_{-i}^\star) \geq u_i(\sigma_i, \sigma_{-i}^\star), \quad \forall \sigma_i \in \Delta(X_i), \tag{6}$$

where $\Delta(X_i) := \{p \in \mathbb{R}^{|X_i|} : 1^{\mathrm{T}} p = 1, p \geq 0\}$.

Theorem 3.5. *Every game $G := \{N, \{X_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N\}$ with X_i being finite for all i has a mixed-strategy Nash equilibrium.*

Proof. The proof follows from the existence of pure-strategy Nash equilibrium. □

Definition 3.6 (Quasi-Concave)**.** A function $f : X \to \mathbb{R}$ is said to be quasi-concave if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda) y) \geq \min\{f(x), f(y)\}.$$

Lemma 3.7. *A function f is quasi-concave if all the level sets of $-f$ are convex.*

Definition 3.8 (Upper-hemicontinuity)**.** Given $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A set-valued function $\psi : X \to Y$ is upper hemicontinuous at $x \in X$ if for any open set $U \subset Y$ such that $\psi(x) \in U$, there exists a $\epsilon > 0$ such that for any $x' \in B_\epsilon(x)$, $\psi(x') \subset U$.

Lemma 3.9. Assume Y is compact. If for all sequence $\{a_n\} \to a \in X$, $b_n \in \psi(a_n)$, $b_n \to b$, we have $b \in \psi(a)$, then ψ is upper hemicontinuous.

Theorem 3.10 (Kakutani Fixed Point Theorem)**.** Suppose $X \subset \mathbb{R}^n$ is nonempty and compact, $\psi : X \to X$ is upper hemicontinuous, and $\psi(x) \subset X$ is nonempty and convex for all $x \in X$. Then $\exists x^\star \in X$ such that $x^\star \in \psi(x^\star)$.

Theorem 3.11 (Existence of Pure-strategy Nash Equilibrium)**.** Consider a normal form game $\{N, \{X_i\}_{i=1}^N, \{\pi_i\}_{i=1}^N\}$. If

1. X_i is nonmepty, convex, and compact,
2. $\pi_i(x_i, x_{-i})$ is continuous in (x_i, x_{-i}) and quasi-concave in x_i,

then, a pure-strategy Nash equilibrium exists.

Proof. Step 1. We first show that for any $x \in \prod_{i=1}^N X_i$, the best response $\mathrm{BR}(x)$ is nonempty. Note that

$$\begin{aligned}\mathrm{BR}(x) &= \mathrm{BR}_1(x_{-1}) \times \mathrm{BR}_2(x_{-2}) \times \cdots \times \mathrm{BR}_N(x_{-N}) \\ &= \operatorname*{argmax}_{x_1} \pi_1(x_1, x_{-1}) \times \operatorname*{argmax}_{x_2} \pi_2(x_2, x_{-2}) \times \cdots \times \operatorname*{argmax}_{x_N} \pi_N(x_N, x_{-N})\end{aligned}$$

For ant $i \in [N]$, according to Weierstrass, $\operatorname{argmax}_{x_i} \pi_i(x_i, x_{-i})$ is nonempty.
Step 2. $\mathrm{BR}(x)$ is convex. It is sufficient to show that for each i, $\mathrm{BR}_i(x_{-i})$ is convex. This is achieved by the fact that π_i is quasi-concave in x_i.
Step 3. $\mathrm{BR}(x)$ is upper hemicontinuous. For any sequence $\{(x_i^n, x_{-i}^n)\} \to (\bar{x}_i, \bar{x}_{-i}) \in \mathrm{BR}(x)$. Let $(y_i^n, y_{-i}^n) \in \mathrm{BR}(x_i^n, x_{-i}^n)$ be such that $(y_i^n, y_{-i}^n) \to (\bar{y}_i, \bar{y}_{-i})$, we know that for any i,

$$y_i^n \in \mathrm{BR}_i(x_{-i}^n) \quad \Rightarrow \quad \pi_i(y_i^n, x_{-i}^n) \geq \pi_i(x_i, x_{-i}^n), \forall x_i \in X_i.$$

Therefore,

$$\pi_i(\bar{y}_i, \bar{x}_{-i}) = \lim_{n\to\infty} \pi_i(y_i^n, x_{-i}^n) \geq \lim_{n\to\infty} \pi_i(x_i, x_{-i}^n) = \pi(x_i, \bar{x}_{-i}), \forall x_i \in X_i, \quad (7)$$

which implies $\bar{y}_i \in \mathrm{BR}_i(\bar{x}_{-i})$. Therefore, $(\bar{y}_i, \bar{y}_{-i}) \in \mathrm{BR}(\bar{x}_i, \bar{x}_{-i})$ and hence $\mathrm{BR}(x)$ is upper-hemicontinuous.

Put step 1,2, and 3 together, we can apply the Kakutani fixed point theorem, which guarantees the existence of $x^\star$ such that $x^\star \in \mathrm{BR}(x^\star)$. $x^\star$ is exactly a pure-strategy Nash equilibrium. □

3.2 Duopolistic Cournot Game

Suppose we have two firms each decides to produce q_1 and q_2 amount of certain product respectively. The market price of the product is $\alpha - \beta(q_1 + q_2)$, where $\alpha, \beta > 0$ are given constants. Assume also the each firm pays $q_i, i = 1, 2$ to produce the product. The optimization problem for the i-th firm is

$$\max_{q_i} \ (\alpha - \beta(q_1 + q_2))q_i - q_i, \tag{8a}$$

$$\text{s.t. } q_i \geq 0. \tag{8b}$$

The KKT condition for the problem is

$$2\beta q_i + \beta q_{-i} - \alpha + 1 = \lambda_i, \tag{9a}$$

$$\lambda_i q_i = 0, \tag{9b}$$

$$\lambda_i \geq 0, \tag{9c}$$

$$q_i \geq 0. \tag{9d}$$

If we stack the conditions for all $i = 1, 2$, we have

$$0 \leq \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \perp \begin{bmatrix} -\alpha + 1 \\ -\alpha + 1 \end{bmatrix} + \begin{bmatrix} 2\beta & \beta \\ \beta & 2\beta \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \geq 0, \tag{10}$$

which can be exactly written in the form

$$0 \leq x \perp g + Mx \geq 0, \tag{11}$$

where $x := [g_1, g_2]^{\mathrm{T}}, g := [-\alpha + 1, -\alpha + 1]^{\mathrm{T}}$, and $M := [2\beta, \beta; \beta, 2\beta]^{\mathrm{T}}$.

3.3 Linear Complementarity Problem

We call the problem of finding a x such that $0 \leq x \perp g + Mx \geq 0$ as linear complementarity problem (LCP) and denote it as LCP(g, M). We call LCP(g, M) is *feasible* if there exists a x_0 such that $x_0 \geq 0$ and $g + Mx_0 \geq 0$. We call it *solvable* if there exists a $x^\star$ such that $0 \leq x^\star \perp g + Mx^\star \geq 0$.

Consider the following quadratic program:

$$\min_{x} \ x^{\mathrm{T}}(g + Mx), \tag{12a}$$

$$\text{s.t. } g + Mx \geq 0, \tag{12b}$$

$$x \geq 0. \tag{12c}$$

It is obvious that if $x^\star$ solves LCP(g, M), then it also solves (12) since $x^{\mathrm{T}}(g + Mx) \geq 0$ by the constraints.

Lemma 3.12. If LCP(g, M) is feasible, then the quadratic program (12) has an optimal solution $z^\star$. Moreover, there exists a vector $u^\star$ of multipliers such that

$$(z^\star - u^\star)_i \left(M^{\mathrm{T}}(z^\star - u^\star)\right)_i \leq 0, \quad \forall i. \tag{13}$$

Proof. Consider the KKT conditions of (12) and let u be the multiplies associated with the constraint $g + Mz \geq 0$. □

The previous Lemma allows us to make an important observation about the solutions of the LCP(g, M) when M is positive semi-definite.

Theorem 3.13. If M is a positive semi-definite matrix of an LCP(g, M) that is feasible, then the LCP(g, M) is solvable.

Proof. By the previous lemma and the positive semi-definiteness of M, we can show

$$(z^\star - u^\star)M^{\mathrm{T}}(z^\star - u^\star) = 0.$$

Simple algebra shows that $(z^\star)^{\mathrm{T}}(g + Mz^\star) = 0$. □

It has been shown that the nature of LCP(g, M) is greatly affected by the structure of the matrix M.

Definition 3.14 (S Matrix)**.** A square matrix $M \in \mathbb{R}^{n \times n}$ is said to be a S *matrix* if there exists a $z > 0$ such that $Mz > 0$.

Proposition 3.15. An LCP(g, M) is feasible for any g if and only if M is a S matrix.

Proof. $(\Rightarrow)$: Let $\bar{g} < 0$. By feasibility there exists a feasible $\tilde{z}$. We know that $M\tilde{z} > 0$. By continuity, there exists $\bar{z} > 0$ such that $M\bar{z} > 0$.
$(\Leftarrow)$: There exists a $\bar{z}$ such that $\bar{z} > 0$ and $M\bar{z} > 0$. For any g, there exists a $\lambda > 0$ such that $M(\lambda\bar{z}) + g \geq 0$ and $\lambda\bar{z} \geq 0$. □

Definition 3.16 (Copositiveness)**.** A matrix $M \in \mathbb{R}^{n \times n}$ is said to be copositive if $x^{\mathrm{T}}Mx \geq 0, \forall x \in \mathbb{R}_n^+$. It is strictly copositive if $x^{\mathrm{T}}Mx > 0$ for all nonzero $x \in \mathbb{R}_n^+$.

Theorem 3.17. If $M \in \mathbb{R}^{n \times n}$ is strictly copositive, then LCP(g, M) is solvable for all g.

Proof. Skipped. □

Next we will consider the uniqueness of solutions to LCP(g, M).

Theorem 3.18. If $M \in \mathbb{R}^{n\times n}$ is positive definite, then LCP(g, M) has a unique solution for all $q \in \mathbb{R}^n$.

Proof. Feasibility follows from the fact that a PD matrix is also a S matrix[2]. Solvability is guaranteed by Theorem 3.17 since PD implies strictly copositiveness. Finally, any solution to LCP(g, M) is a solution to the QP (12) by Lemma 3.12. Since the QP is strictly convex. The uniqueness of solution to LCP(g, M) is proved. □

Definition 3.19. A matrix $M \in \mathbb{R}^{n\times n}$ is a P matrix if the principal minors are all positive.

Remark 3.20. A PD matrix is a P matrix but the converse is in general not true. As an example, consider the matrix $M := \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$. All principal minors are positive but it is not PD.

Theorem 3.21. If $M \in \mathbb{R}^{n\times n}$ is a P matrix, then LCP(g, M) has a unique solution.

Consider the duopolistic Cournot game with productivity cobstraint $0 \leq q_i \leq K_i$ for some $K_i > 0$. Stack all the KKT conditions gives

$$0 \leq \begin{bmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \perp \begin{bmatrix} -\alpha + 1 \\ -\alpha + 1 \\ K_1 \\ K_2 \end{bmatrix} + \begin{bmatrix} 2\beta & \beta & 1 & 0 \\ \beta & 2\beta & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \geq 0. \tag{14}$$

Note that M is not PD but PSD. Moreover, M is neither a S matrix nor a P matrix. Therefore, none of our results established so far can guarantee the uniqueness of LCP(g, M), even the existence is not guaranteed.

[2]To prove the fact, one needs to apply the *Ville's theorem*: Consider the below two systems of equations:

$$\begin{cases} Ax > 0, \\ x > 0 \end{cases} \quad \begin{cases} A^{\mathrm{T}} y \leq 0, \\ y \leq 0. \end{cases}$$

Either there exists a solution to the first system or there exists a nontrivial solution to the second system.

Lemma 3.22 (ω-Uniqueness)**.** Let $M \in \mathbb{R}^{n\times n}$ be PSD and g be arbitrary. Then,

1. If $z_1, z_2 \in \mathrm{SOL}(g, M)$ then
$$z_1^{\mathrm{T}}(g + Mz_2) = z_2^{\mathrm{T}}(g + Mz_1) = 0.$$
2. If in addition that M is symmetric, then $Mz_1 = Mz_2, \forall z_1, z_2 \in \mathrm{SOL}(g, M)$.

3.4 Variational Inequality

As a motivating example, consider again the cournot game. Now, we assume the price to be $\alpha \exp(-\beta \sum_{j=1}^{2} q_j)$, where $\alpha, \beta > 0$.

Stacking the KKT conditions now gives

$$0 \leq \begin{bmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \perp \begin{bmatrix} \lambda_1 + 1 - \alpha \exp(-\beta(q_1 + q_2))(1 - \beta_1 q_1) \\ \lambda_2 + 1 - \alpha \exp(-\beta(q_1 + q_2))(1 - \beta_2 q_2) \\ K_1 - q_1 \\ K_2 - q_2 \end{bmatrix} \geq 0, \tag{15}$$

which is indeed a non-linear complementarity problem.

Note also that the optimization problem is a convex problem, we can directly write down the optimality condition.

$$(y_i - q_i)^{\mathrm{T}} \left(\alpha e^{-\beta(q_1+q_2)} + \alpha\beta q_1 e^{-\beta(q_1+q_2)} + 1\right) \leq 0, \quad \forall 0 \leq q_i \leq K_i, i = 1, 2. \tag{16}$$

Stacking (16) for $i = 1, 2$ yields

$$\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right)^{\mathrm{T}} \begin{bmatrix} \alpha e^{-\beta(q_1+q_2)} + \alpha\beta q_1 e^{-\beta(q_1+q_2)} + 1 \\ \alpha e^{-\beta(q_1+q_2)} + \alpha\beta q_2 e^{-\beta(q_1+q_2)} + 1 \end{bmatrix} \leq 0, \forall \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in [0, K_1] \times [0, K_2]. \tag{17}$$

Definition 3.23 (Variational Inequality)**.** Variational inequaility (VI), $\mathrm{VI}(K, F)$, where $F : K \subset \mathbb{R}^n \to \mathbb{R}^n$ is to find $x^\star \in K$ such that

$$(y - x^\star)^{\mathrm{T}} F(x^\star) \geq 0, \quad \forall y \in K. \tag{18}$$

Definition 3.24. Let K be a *cone*. Complementarity problem (CP) $\mathrm{CP}(K, F)$, where $F : K \subset \mathbb{R}^n \to \mathbb{R}^n$ is to find a $x^\star \in K$ such that $x^\star \in K$, $F(x^\star) \in K^D$, and

$$x^\star \perp F(x^\star), \tag{19}$$

where $K^D := \{v : v^{\mathrm{T}} x \geq 0, \forall x \in K\}$ is the dual cone of K.

Remark 3.25. (VI) does not assume any structure on K but (CP) does assume it to be a cone.

Lemma 3.26. If K is a cone, then (CP) and (VI) are equivalent.

Proof. ($\Rightarrow$): Suppose x solves $\text{VI}(K, F)$, then for $\forall y \in K$, $(y-x)^{\mathrm{T}}F(x) \geq 0$. Since K is a cone, $0 \in K$, then $(0-x)^{\mathrm{T}}F(x) \geq 0$, which implies that $x^{\mathrm{T}}F(x) \leq 0$. Also, $2x \in K$ since K is a cone, we thus have $x^{\mathrm{T}}F(x) \geq 0$ and therefore $x^{\mathrm{T}}F(x) = 0$. Note that $(y-x)^{\mathrm{T}}F(x) \geq 0, \forall y \in K$ implies $y^{\mathrm{T}}F(x) \geq 0, \forall y \in K$. Therefore, $F(x) \in K^D$.

The reverse implication is obvious. □

3.5 Some Special CPs

Example 3.27. $K = \mathbb{R}^n_+$: In this case, $K^D = \mathbb{R}^+_n$. Therefore, $\text{CP}(K, F)$ becomes $x \leq x \perp F(x) \geq 0$.

Example 3.28. $K = \mathbb{R}^n$: In this case, $K^D = \{0\}$. $\text{CP}(K, F)$ is equivalent to $F(x) = 0$.

Remark 3.29. The above example shows that solving system of nonlinear equations is a special instance of complementarity problems.

Example 3.30. $K = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_+$: In this case, $CP(K, F)$ is equivalent to

$$G(u, v) = 0, \tag{20a}$$

$$0 \leq v \perp H(u, v) \geq 0, \tag{20b}$$

where $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_+ \to \mathbb{R}^{n_1}$ and $H : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_+ \to \mathbb{R}^{n_2}$. The complementarity problem is called Mixed complementarity problem (MiCP), which represents the KKT conditions for some optimization problem with both equality and inequality constraints.

3.6 Solution Analysis of VI and CP

Theorem 3.31 (Existence)**.** Let $K \subset \mathbb{R}^n$ be nonempty, convex, and compact and let $F : K \to \mathbb{R}^n$ be continuous. Then $\text{SOL}(K, F) \neq \emptyset$ and is compact.

Remark 3.32. A cone can be compact (bounded and closed in finite-dimensional spaces). The only example is $K = \{0\}$.

Definition 3.33 (Monotonicity)**.** A function $F : K \subset \mathbb{R}^n \to \mathbb{R}^n$ is

(a). *psuedo-monotone* on K if $\forall x, y \in K$, $(x-y)^{\mathrm{T}}F(y) \geq 0$ implies $(x-y)^{\mathrm{T}}F(x) \geq 0$.

(b). *monotone* on K if $(F(x) - F(y))^{\mathrm{T}}(x-y) \geq 0, \forall x, y \in K$.

(c). *strictly monotone* on K if $(F(x) - F(y))^{\mathrm{T}}(x-y) > 0, \forall x, y \in K$.

(d). *ϵ-montone* on K for some $\epsilon > 1$, if $\exists C > 0$ such that $(F(x) - F(y))^{\mathrm{T}}(x-y) \geq C\|x-y\|^{\epsilon}, \forall x, y \in K$.

(e). *strongly monotone* on K if $\exists C > 0$ such that $(F(x)-F(y))^{\mathrm{T}}(x-y) \geq C\|x-y\|^2, \forall x, y \in K$.

Definition 3.34 (Jacobian)**.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$. The *Jacobian* of F is defined to be

$$JF(x) := \begin{bmatrix} \nabla_x f_1(x_1, ..., x_n)^{\mathrm{T}} \\ \vdots \\ \nabla_x f_n(x_1, ..., x_n)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}. \tag{21}$$

Remark 3.35. The Jacobian for an arbitrary function F in general is not symmetric.

Theorem 3.36. If $JF(x) \in \mathbb{R}^{n \times n}$ is symmetric, then $\mathrm{CP}(K, F)$ must be KKT conditions for some optimization problem.

Proposition 3.37. Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on a convex set D.

(a). F is monotone on D if and only if $JF(x)$ is PSD, $\forall x \in D$.

(b). F is strictly monotone on D if $JF(x)$ is PD, $\forall x \in D$.

(c). F is strongly monotone on D if and only if $JF(x)$ is uniformly positive definite ($\exists C' > 0$ such that $y^{\mathrm{T}}JF(x)y \geq C'\|y\|^2, \forall y \in \mathbb{R}^n$), $\forall x \in D$.

Theorem 3.38 (Existence and Uniqueness)**.** Let $F : K \subset \mathbb{R}^n \to \mathbb{R}^n$,

(a). $\mathrm{VI}(K, F)$ has at most one solution if F is strictly monotone.

(b). $\mathrm{VI}(K, F)$ has a unique solution if F is ϵ-monotone on K.

3.7 Dynamic Games

Example 3.39. Two players. Player 1 moves first. Player 2 observes player 1's action and then make a deicision. The payoff functions for the two players are

$$\pi_i(q_1, q_2) := (a - (q_1 + q_2))q_i - cq_i, \quad i = 1, 2, \tag{22}$$

where $c_i \in \mathbb{R}$. Suppose player 1's action is $q_1 \geq 0$. Then, the best response of player 2 to q_1 is

$$\mathrm{BR}_2(q_1) \in \operatorname*{argmax}_{q_2 \geq 0} (a - (q_1 + q_2))q_2 - cq_2. \tag{23}$$

Assume $q_2 > 0$, $q_2^\star(q_1) = \frac{a-c-q_1}{2}$. Therefore, Knowing that player 2 is always going to play $q_2^\star$, player 1 will choose a $q_1 \in \operatorname{argmax}_{q_1 \geq 0} \pi_2(q_1, q_2^\star(q_1))$. KKT condition to the optimization problem is

$$0 \leq q_1 \perp -(a - q_1 - \frac{1}{2}(a - c - q_1) - \frac{1}{2}q_1 - c) \geq 0. \tag{24}$$

Assume $q_1 > 0$, we have $q_1^\star = \frac{1}{2}(a-c)$ and thus $q_2^\star = \frac{a-c}{4}$, which give optimal payoffs $\pi_1^\star = \frac{1}{8}(a-c)^2$ and $\frac{1}{16}(a-c)^2$, for player 1 and 2 respectively.

If two players simultaneously move, their optimal decisions are given by solving the system of optimization problems:

$$\max_{q_i \geq 0} \pi_i(q_i, q_{-i}), \quad i = 1, 2. \tag{25}$$

Stacking KKT conditions for the two problems yield

$$\mathbf{0} \leq \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \perp \begin{bmatrix} -(a - c - 2q_1 - q_2) \\ -(a - c - 2q_2 - q_1) \end{bmatrix} \geq \mathbf{0}. \tag{26}$$

Assume $q_1, q_2 > 0$. By symmetry of the game it is not hard to see that the optimal decisions $q_1^\star = q_2^\star = \frac{a-c}{3}$. The optimal payoff is $\frac{1}{9}(a-c)^2$, which is the same for both of them.

Compared with the sequential move case, player 1 obtains a higher payoff. Although player 2 knows more, she (he) achieves a lower payoff. This is exactly the so-called **First-move Advantange**.

Note that the action profile $(\frac{a-c}{2}, \frac{a-c}{4})$ is called the **Subgame Perfect Nash Equilibrium (SPNE)** of the game. It is also a **Nash equilibrium** since given that both players are playing this profile, no one will have any incentive to unilaterally deviate. Therefore, *the set of subgame perfect Nash equilibrium is a subset of the set of Nash equilibrium.* The converse is not true. $(\frac{a-c}{3}, \frac{a-c}{3})$ is a Nash equilibrium but not a subgame perfect Nash equilibrium since player 1 can deviate to obtain a higher payoff.

Definition 3.40 (Extensive Form)**.** The extensive form of a game contains the following elements: (1). The set of players; (2). when does each player move; what can each player do at their turn; what does each player knwo at their turn; (3). payoffs are specified at the leaves.

Definition 3.41 (Information Set)**.** Information set is a collection of nodes satisfying:

(1). A player has to move at each node in the information set.

(2). When the play of the game reaches a node in the information set, the player with the same move does not know which node in the information set the game reaches.